NATURAL BOUNDARY CONDITIONS IN THE CALCULUS OF VARIATIONS

Agnieszka B. Malinowska

Department of Mathematics, University of Aveiro
3810-193 Aveiro, Portugal

Delfim F. M. Torres

Department of Mathematics, University of Aveiro
3810-193 Aveiro, Portugal

Abstract. We prove necessary optimality conditions for problems of the calculus of variations on time scales with a Lagrangian depending on the free end-point.

1. **Introduction.** The calculus on time scales was introduced by Bernd Aulbach and Stefan Hilger in 1988 [7]. The new theory unify and extends the traditional areas of continuous and discrete analysis and the various dialects of q-calculus [14] into a single theory [13, 24], and is finding numerous applications in such areas as engineering, biology, economics, finance, and physics [1]. The present work is dedicated to the study of problems of calculus of variations on a generic time scale $\mathbb{T}$. As particular cases, one gets the classical calculus of variations [17] by choosing $\mathbb{T} = \mathbb{R}$; the discrete-time calculus of variations [23] by choosing $\mathbb{T} = \mathbb{Z}$; and the q-calculus of variations [8] by choosing $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$, $q > 1$.

The calculus of variations on time scales was born with the works [3] and [10] and seems to have interesting applications in Economics [4, 5, 6, 15]. Currently, several researchers are getting interested in the new theory and contributing to its development (see, e.g., [9, 11, 16, 19, 25]). Here we develop further the theory by proving necessary optimality conditions for more general problems of the calculus of variations with a Lagrangian that may also depend on the unspecified end-point $x(T)$.

In Section 2 we review the necessary concepts and tools on time scales; our results are given in Section 3. We begin Section 3 by formulating the problem (1)–(2) under study: to minimize a delta-integral functional subject to a given fixed initial-point $x(a) = \alpha$ and having no constraint on $x(T)$. The novelty is the dependence of the integrand f on the free end-point $x(T)$. Necessary optimality conditions for such problems, on a general time scale, are given using both Lagrangian (Theorem 3.2) and Hamiltonian formalisms (Theorem 3.4). Under appropriate convexity and linearity assumptions, the conditions turn out to be sufficient for a global minimum (cf.

2000 *Mathematics Subject Classification.* Primary: 49K05, 39A12; Secondary: 49-99, 93-99.

Key words and phrases. Calculus of variations, optimal control, time scales, discrete-time, transversality conditions.

The first author is on leave of absence from Faculty of Computer Science, Białystok Technical University, 15-351 Białystok, Poland. E-mail: abmalina@pb.bialystok.pl.

Theorem 3.5). A number of important corollaries are obtained, and several examples illustrating the new results discussed in detail. Corollary 1 (see also Corollary 4) give answer to a question posed to the second author by A. Zinober in May 2008 during a visit to the University of Aveiro, and again presented as an open question during the conference "Calculus of Variations and Applications—from Engineering to Economy", held from 8th to 10th September 2008 in the New University of Lisbon, Monte de Caparica, Portugal: "What are the necessary optimality conditions for the problem of the calculus of variations with a free end-point $x(T)$ but whose Lagrangian depends explicitly on $x(T)$?" The new transversality condition (37) (or the equivalent natural boundary condition (10)) seems to have important implications in Economics. This question is under study by Alan Zinober, Kim Kaivanto, and Pedro Cruz and will appear elsewhere.

2. **Preliminaries.** In this section we introduce basic definitions and results that will be needed for the rest of the paper. For a more general presentation of the theory of time scales, we refer the reader to [12].

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and it is denoted by $\mathbb{T}$. Thus, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, are trivial examples of times scales. Other examples of times scales are: $[-2,4]\bigcup\mathbb{N}$, $h\mathbb{Z} := \{hz | z \in \mathbb{Z}\}$ for some $h > 0$, $q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$ for some $q > 1$, and the Cantor set. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively.

The *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

Example 1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\rho(t) = t-1$, and $\mu(t) = 1$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a fixed real number, then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, and $\mu(t) = (q-1)t$.

Definition 2.1. [12] A time scale $\mathbb{T}$ is called *regular* if the following two conditions are satisfied simultaneously:

(i) $\sigma(\rho(t)) = t$, for all $t \in \mathbb{T}$;
(ii) $\rho(\sigma((t)) = t$, for all $t \in \mathbb{T}$.

Following [12], let us define $\mathbb{T}^\kappa = \mathbb{T}\backslash(\rho(b), b]$.

Definition 2.2. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t and say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Remark 1. If $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$, then $f^\Delta(t)$ is not uniquely defined, since for such a point t, small neighborhoods U of t consist only of t and, besides, we have $\sigma(t) = t$. For this reason, maximal left-scattered points are omitted in Definition 2.2.

Note that in right-dense points $f^\Delta(t) = lim_{s \to t} = \frac{f(t)-f(s)}{t-s}$, provided this limit exists, and in right-scattered points $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$, provided f is continuous at t.

Example 2. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$, i.e., the delta derivative coincides with the usual one. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$, i.e., we get the usual derivative of Quantum calculus [22].

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by C_{rd} and the set of all delta differentiable functions with rd-continuous derivative by $\mathrm{C}^1_{\mathrm{rd}}$. It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function F with $F^\Delta = f$, and in this case the *delta integral* is defined by $\int_c^d f(t)\Delta t = F(d) - F(c)$ for all $c, d \in \mathbb{T}$.

Example 3. Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the classical Riemann integral. If $\mathbb{T} = \mathbb{Z}$, then $\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\int_a^b f(t)\Delta t = (1-q)\sum_{t\in[a,b)} tf(t)$.

The delta integral has the following properties:

(i) if $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then
$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)\,;$$

(ii) if $c, d \in \mathbb{T}$ and $f, g \in C_{rd}$, then
$$\int_c^d f(\sigma(t))g^\Delta(t)\Delta t = [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\Delta(t)g(t)\Delta t\,,$$
$$\int_c^d f(t)g^\Delta(t)\Delta t = [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\Delta(t)g(\sigma(t))\Delta t.$$

The Dubois-Reymond lemma of the calculus of variations on time scales will be useful for our purposes.

Lemma 2.3. *(Lemma of Dubois-Reymond [10]) Let $g \in C_{rd}$, $g : [a,b]^k \to \mathbb{R}$. Then,*
$$\int_a^b g(t)\cdot\eta^\Delta(t)\Delta t = 0 \quad \textit{for all } \eta \in C^1_{rd} \textit{ with } \eta(a) = \eta(b) = 0$$
if and only if $g(t) = c$ on $[a,b]^k$ for some $c \in \mathbb{R}$.

3. **Main Results.** Let $\mathbb{T}$ be a bounded time scale. Throughout we let $A, B \in \mathbb{T}$ with $A < B$. For an interval $[c,d] \cap \mathbb{T}$ we simply write $[c,d]$. We also abbreviate $f \circ \sigma$ by f^σ. Now let $[a,T]$ with $T < B$ be a subinterval of $[A,B]$. The problem of the calculus of variations on time scales under consideration has the form
$$\text{minimize} \quad \mathcal{L}[x] = \int_a^T f(t, x^\sigma(t), x^\Delta(t), x(T))\Delta t, \tag{1}$$

over all $x \in C^1_{rd}$ satisfying the boundary condition

$$x(a) = \alpha\,, \quad \alpha \in \mathbb{R} \quad (x(T) \text{ free}), \tag{2}$$

where the function $(t,x,v,z) \to f(t,x,v,z)$ from $[a,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ has partial continuous derivatives with respect to x, v, z for all $t \in [a,T]$, and $f(t,\cdot,\cdot,\cdot)$ and its partial derivatives are rd-continuous for all t. A function $x \in C^1_{rd}$ is said to be admissible if it is satisfies condition (2).

Let us consider the following norm in C^1_{rd}:

$$\|x\|_1 = sup_{t\in[a,T]}|x^\sigma(t)| + sup_{t\in[a,T]}|x^\Delta(t)|.$$

Definition 3.1. An admissible function $\tilde{x}$ is said to be a *weak local minimum* for (1)–(2) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \le \mathcal{L}[x]$ for all admissible x with $\|x - \tilde{x}\|_1 < \delta$.

3.1. **Lagrangian approach.** Next theorem gives necessary optimality conditions for problem (1)–(2).

Theorem 3.2. *If $\tilde{x}(\cdot)$ is a solution of the problem (1)–(2), then*

$$f^\Delta_{x^\Delta}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t),\tilde{x}(T)) = f_{x^\sigma}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t),\tilde{x}(T)) \tag{3}$$

for all $t \in [a,T]^\kappa$. Moreover,

$$f_{x^\Delta}(\rho(T),\tilde{x}^\sigma(\rho(T)),\tilde{x}^\Delta(\rho(T)),\tilde{x}(T)) + \int_{\rho(T)}^T f_{x^\sigma}(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t),\tilde{x}(T))\Delta t \\ + \int_a^T f_z(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t),\tilde{x}(T))\Delta t = 0. \tag{4}$$

Proof. Suppose that $\mathcal{L}[\cdot]$ has a weak local minimum at $\tilde{x}(\cdot)$. We can proceed as Lagrange did, by considering the value of $\mathcal{L}$ at a nearby function $x = \tilde{x} + \varepsilon h$, where $\varepsilon \in \mathbb{R}$ is a small parameter, $h(\cdot) \in C^1_{rd}$, and $h(a) = 0$. Because $x(T)$ is free, we do not require $h(\cdot)$ to vanish at T. Let

$$\phi(\varepsilon) = \mathcal{L}[(\tilde{x}+\varepsilon h)(\cdot)] = \int_a^T f(t,\tilde{x}^\sigma(t)+\varepsilon h^\sigma(t),\tilde{x}^\Delta(t)+\varepsilon h^\Delta(t),\tilde{x}(T)+\varepsilon h(T))\Delta t.$$

A necessary condition for $\tilde{x}(\cdot)$ to be a minimum is given by

$$\phi'(\varepsilon)|_{\varepsilon=0} = 0 \Leftrightarrow \int_a^T \left[f_{x^\sigma}(\cdots)h^\sigma(t) + f_{x^\Delta}(\cdots)h^\Delta(t) + f_z(\cdots)h(T)\right]\Delta t = 0\,, \tag{5}$$

where $(\cdots) = \left(t,\tilde{x}^\sigma(t),\tilde{x}^\Delta(t),\tilde{x}(T)\right)$. Integration by parts gives

$$\int_a^T f_{x^\sigma}(\cdots)h^\sigma(t)\Delta t = \int_a^t f_{x^\sigma}(\cdots)\Delta\tau h(t)|_{t=a}^{t=T} - \int_a^T \left(\int_a^t f_{x^\sigma}(\cdots)\Delta\tau h^\Delta(t)\right)\Delta t.$$

Because $h(a) = 0$, the necessary condition (5) can be written as

$$0 = \int_a^T \left(f_{x^\Delta}(\cdots) - \int_a^t f_{x^\sigma}(\cdots)\Delta\tau\right)h^\Delta(t)\Delta t \\ + \int_a^T f_{x^\sigma}(\cdots)\Delta\tau h(T) + \int_a^T f_z(\cdots)\Delta t h(T) \tag{6}$$

for all $h(\cdot) \in C_{rd}^1$ such that $h(a) = 0$. In particular, equation (6) holds for the subclass of functions $h(\cdot) \in C_{rd}^1$ that do vanish at $h(T)$. Thus, by the Dubois-Reymond Lemma 2.3, we have

$$f_{x^\Delta}(\cdots) - \int_a^t f_{x^\sigma}(\cdots)\Delta\tau = c, \tag{7}$$

for some $c \in \mathbb{R}$ and all $t \in [a, T]$. Equation (6) must be satisfied for all $h(\cdot) \in C_{rd}^1$ with $h(a) = 0$, which includes functions $h(\cdot)$ that do not vanish at T. Consequently, equations (6) and (7) imply that

$$c + \int_a^T f_{x^\sigma}(\cdots)\Delta t + \int_a^T f_z(\cdots)\Delta t = 0. \tag{8}$$

From the properties of the delta integral and from (7), it follows that

$$\begin{aligned} c + \int_a^T f_{x^\sigma}(\cdots)\Delta t &= c + \int_a^{\rho(T)} f_{x^\sigma}(\cdots)\Delta t + \int_{\rho(T)}^T f_{x^\sigma}(\cdots)\Delta t \\ &= f_{x^\Delta}(\rho(T), \tilde{x}^\sigma(\rho(T)), \tilde{x}^\Delta(\rho(T)), \tilde{x}(T)) + \int_{\rho(T)}^T f_{x^\sigma}(\cdots)\Delta t\,. \end{aligned}$$

Hence, we can rewrite (8) as (4). □

Theorem 3.3. *Let $\mathbb{T}$ be a regular time scale. If $\tilde{x}(\cdot)$ is a solution of the problem (1)–(2), then*

$$f_{x^\Delta}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t), \tilde{x}(T)) = f_{x^\sigma}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t), \tilde{x}(T))$$

for all $t \in [a, T]^\kappa$. Moreover,

$$\begin{aligned} f_{x^\Delta}(\rho(T), \tilde{x}^\sigma(\rho(T)), \tilde{x}^\Delta(\rho(T)), \tilde{x}(T)) \\ + \mu(\rho(T)) f_{x^\sigma}(\rho(T)), \tilde{x}^\sigma(\rho(T)), \tilde{x}^\Delta(\rho(T)), \tilde{x}(T)) \\ + \int_a^T f_z(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t), \tilde{x}(T))\Delta t = 0. \end{aligned} \tag{9}$$

Proof. By Theorem 3.2 we need only to show that on a regular time scale equation (4) can be written in the form (9). Indeed, from the properties of the delta integral it follows that

$$\int_{\rho(T)}^T f_{x^\sigma}(\cdots)\Delta t = \mu(\rho(T)) f_{x^\sigma}(\rho(T)), \tilde{x}^\sigma(\rho(T)), \tilde{x}^\Delta(\rho(T)), \tilde{x}(T))\,.$$

□

Choosing $\mathbb{T} = \mathbb{R}$ in Theorem 3.3 we immediately obtain the corresponding result in the classical context of the calculus of variations. We were not able to find a reference, in the vast and rich literature of the calculus of variations, to the result given by Corollary 1.

Corollary 1. *If $\tilde{x}(\cdot)$ is a solution of the problem*

$$\text{minimize} \quad \mathcal{L}[x] = \int_a^T f(t, x(t), x'(t), x(T))dt$$
$$x(a) = \alpha\,, \quad \alpha \in \mathbb{R} \quad (x(T) \text{ free}),$$

where $a, T \in \mathbb{R}$, $a < T$, $x(\cdot) \in C^1$, *then the Euler-Lagrange equation*

$$\frac{d}{dt} f_{x'}\left(t, \tilde{x}(t), \tilde{x}'(t), \tilde{x}(T)\right) = f_x\left(t, \tilde{x}(t), \tilde{x}'(t), \tilde{x}(T)\right)$$

holds for all $t \in [a, T]$. *Moreover,*

$$f_{x'}\left(T, \tilde{x}(T), \tilde{x}'(T), \tilde{x}(T)\right) = -\int_a^T f_z\left(t, \tilde{x}(t), \tilde{x}'(t), \tilde{x}(T)\right) dt\,. \tag{10}$$

Remark 2. In the classical setting f does not depend on $x(T)$, i.e., $f_z = 0$. Then, (10) reduces to the well known natural boundary condition $f_{x'}\left(T, \tilde{x}(T), \tilde{x}'(T)\right) = 0$.

Similarly, we can obtain other corollaries by choosing different time scales. Corollary 2 is obtained from Theorem 3.3 letting $\mathbb{T} = \mathbb{Z}$.

Corollary 2. *If* $\tilde{x}(\cdot)$ *is a solution of the discrete-time problem*

$$\begin{gathered} \textit{minimize} \quad \mathcal{L}[x] = \sum_{t=a}^{T-1} f(t, x(t+1), \Delta x(t), x(T)) \\ x(a) = \alpha\,, \quad \alpha \in \mathbb{R} \quad (x(T)\ \textit{free}), \end{gathered}$$

where $a, T \in \mathbb{Z}$, $a < T$, *then*

$$f_x\left(t, \tilde{x}(t+1), \Delta\tilde{x}(t), \tilde{x}(T)\right) = \Delta f_{\Delta x}\left(t, \tilde{x}(t+1), \Delta\tilde{x}(t), \tilde{x}(T)\right)$$

for all $t \in [a, T-1]$. *Moreover,*

$$\begin{aligned} f_x\left(T-1, \tilde{x}(T), \Delta\tilde{x}(T-1), \tilde{x}(T)\right) + f_{\Delta x}\left(T-1, \tilde{x}(T), \Delta\tilde{x}(T-1), \tilde{x}(T)\right) \\ = -\sum_{t=a}^{T-1} f_z(t, \tilde{x}(t+1), \Delta\tilde{x}(t), \tilde{x}(T))\,. \end{aligned} \tag{11}$$

Remark 3. In the case f does not depend on $x(T)$, (11) reduces to the natural boundary condition for the discrete variational problem (see [23, Theorem 8.3]).

Let now $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. We then obtain the analogous result for the q-calculus of variations. In what follows we use the standard notation D_q for the q-derivative:

$$D_q x(t) = \frac{x(qt) - x(t)}{qt - t} \tag{12}$$

(cf. Example 2). The q-derivative (12) is also known in the literature as the Jackson derivative [21].

Corollary 3. *If* $\tilde{x}(\cdot)$ *is a solution of the problem*

$$\begin{gathered} \textit{minimize} \quad \mathcal{L}[x] = \sum_{t=a}^{Tq^{-1}} (q-1)t f\left(t, x(qt), D_q x(t), x(T)\right) \\ x(a) = \alpha\,, \quad \alpha \in \mathbb{R} \quad (x(T)\ \textit{free})\,, \end{gathered}$$

where $a, T \in \mathbb{T}$, $a < T$, *then*

$$f_x\left(t, \tilde{x}(qt), D_q\tilde{x}(t), \tilde{x}(T)\right) = D_q f_v\left(t, \tilde{x}(qt), D_q\tilde{x}(t), \tilde{x}(T)\right)$$

for all $t \in \left[a, Tq^{-1}\right]$*. Moreover,*

$$\begin{aligned} f_v &\left(Tq^{-1}, \tilde{x}(T), D_q\tilde{x}\left(Tq^{-1}\right), \tilde{x}(T)\right) \\ &\qquad + T(1-q^{-1}) f_x\left(Tq^{-1}, \tilde{x}(T), D_q\tilde{x}\left(Tq^{-1}\right), \tilde{x}(T)\right) \\ &\qquad\qquad = -\sum_{t=a}^{Tq^{-1}} f_z\left(t, \tilde{x}(qt), D_q\tilde{x}(t), \tilde{x}(T)\right). \end{aligned}$$

We illustrate the application of our Theorem 3.2 with an example.

Example 4. Consider the problem

$$\text{minimize} \quad \mathcal{L}[x] = \int_0^1 \left(\sqrt{1 + (x^\Delta(t))^2} + \beta(x(1)-1)^2\right)\Delta t, \tag{13}$$

where $\beta \in \mathbb{R}^+$, subject to the boundary condition

$$x(0) = 0 \quad (x(1) \text{ free}). \tag{14}$$

Since

$$f(t, x^\sigma, x^\Delta, z) = \sqrt{1 + (x^\Delta)^2} + \beta(z-1)^2,$$

we have

$$\begin{aligned} f_{x^\sigma}(t, x^\sigma, x^\Delta, z) &= 0, \\ f_{x^\Delta}(t, x^\sigma, x^\Delta, z) &= \frac{x^\Delta}{\sqrt{1+(x^\Delta)^2}}, \\ f_z(t, x^\sigma, x^\Delta, z) &= 2\beta(z-1). \end{aligned}$$

If $\tilde{x}(\cdot)$ is a local minimizer of (13)–(14), then conditions (3)–(4) must hold, i.e.,

$$f^\Delta_{x^\Delta}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t), \tilde{x}(T)) = 0, \tag{15}$$

$$f_{x^\Delta}(\rho(1), \tilde{x}^\sigma(\rho(1)), \tilde{x}^\Delta(\rho(1)), \tilde{x}(1)) = -\int_0^1 2\beta(\tilde{x}(1)-1)\Delta t. \tag{16}$$

Equation (15) implies that there exists a constant $d \in \mathbb{R}$ such that

$$\tilde{x}^\Delta(t) = d\sqrt{1 + (\tilde{x}^\Delta(t))^2}.$$

Solving the latter equation with initial condition $\tilde{x}(0) = 0$ we obtain $\tilde{x}(t) = \alpha t$, where $\alpha \in \mathbb{R}$. In order to determine α we use the natural boundary condition (16), which can be rewritten as

$$\frac{\alpha}{\sqrt{1+\alpha^2}} = -2\beta(\alpha - 1). \tag{17}$$

The real solution of equation (17) is

$$\alpha(\beta) = \frac{2\,\beta^2 + \sqrt{4\,\beta^4 + \beta^2}}{4\beta^2} - \frac{\sqrt{-8\,\beta^2 + 4\,\sqrt{4\,\beta^4+\beta^2} + 1}}{4\beta}.$$

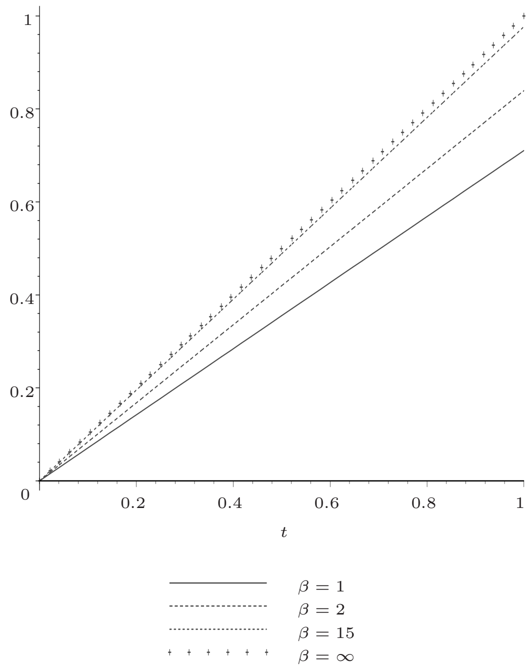

FIGURE 1. The extremal $\tilde{x}(t) = \alpha(\beta)t$ of Example 4 for different values of the parameter β.

Hence, $\tilde{x}(t) = \alpha(\beta)t$ is a candidate to be a minimizer with

$$\mathcal{L}[\tilde{x}] = \sqrt{1 + \left(\frac{2\,\beta^2 + \sqrt{4\,\beta^4 + \beta^2}}{4\beta^2} - \frac{\sqrt{-8\,\beta^2 + 4\,\sqrt{4\,\beta^4 + \beta^2} + 1}}{4\beta}\right)^2} + \beta\left(\frac{2\,\beta^2 + \sqrt{4\,\beta^4 + \beta^2}}{4\beta^2} - \frac{\sqrt{-8\,\beta^2 + 4\,\sqrt{4\,\beta^4 + \beta^2} + 1}}{4\beta} - 1\right)^2 .$$

The extremal $\tilde{x}(t) = \alpha(\beta)t$ is represented in Figure 4 for different values of β. We note that

$$\lim_{\beta\to\infty} \alpha(\beta) = 1,$$

and in the limit, when $\beta = \infty$, the solution of (13)–(14) coincides with the solution of the following problem with fixed initial and terminal points (cf. [10]):

$$\begin{gathered}\text{minimize} \quad \mathcal{L}[x] = \int_0^1 \left(\sqrt{1 + (x^\Delta(t))^2}\right)\Delta t \\ x(0) = 0\,, \quad x(1) = 1\,.\end{gathered}$$

3.2. **Hamiltonian approach.** Hamiltonian systems on time scales were introduced in [2] and have a central role in the study of optimal control problems on time scales [20]. Let us consider now the more general problem

$$\text{minimize} \quad \mathcal{L}[x,u] = \int_a^T f(t, x^\sigma(t), u^\sigma(t), x(T))\Delta t \tag{18}$$

subject to

$$x^\Delta(t) = g(t, x^\sigma(t), u^\sigma(t), x(T)), \tag{19}$$

$$x(a) = \alpha\,, \quad \alpha \in \mathbb{R} \quad (x(T) \text{ free}), \tag{20}$$

where $f(t,x,v,z) : [a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g(t,x,v,z) : [a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ have partial continuous derivatives with respect to x, v, z for all $t \in [a,T]$, and $f(t,\cdot,\cdot,\cdot)$, $g(t,\cdot,\cdot,\cdot)$ and their partial derivatives are rd-continuous for all t. In the particular case $g(t,x,v,z) = v$ problem (18)–(20) reduces to (1)–(2).

Theorem 3.4. *If $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is a weak local minimizer for the problem (18)–(20), then there is a function $\tilde{\lambda}(\cdot)$ such that the triple $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{\lambda}(\cdot))$ satisfies: (i) the Hamiltonian system*

$$x^\Delta(t) = H_{\lambda^\sigma}(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T)), \tag{21}$$

$$(\lambda^\sigma(t))^\Delta = -H_{x^\sigma}(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T)), \tag{22}$$

(ii) the stationary condition

$$H_{u^\sigma}(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T)) = 0, \tag{23}$$

for all $t \in [a,T]^\kappa$; and (iii) the transversality condition

$$\lambda^\sigma(\rho(T)) = \int_{\rho(T)}^T H_{x^\sigma}(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T))\Delta t + \int_a^T H_z(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T))\Delta t, \tag{24}$$

where the Hamiltonian $H(t,x,v,\lambda,z) : [a,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(t, x^\sigma, u^\sigma, \lambda^\sigma, z) = f(t, x^\sigma, u^\sigma, z) + \lambda^\sigma g(t, x^\sigma, u^\sigma, z). \tag{25}$$

Remark 4. In Theorem 3.4 we are assuming to have a time scale $\mathbb{T}$ for which $\lambda^\sigma(t)$ is delta-differentiable on $[a,T]^\kappa$. Examples of time scales for which σ is not delta-differentiable are easily found [12].

Proof. Using the Lagrange multiplier rule we can form an expression $\lambda^\sigma(g - x^\Delta)$ for each value of t. The replacement of f by $f + \lambda^\sigma(g - x^\Delta)$ in the objective functional give us the following new problem:

$$\text{minimize} \quad \mathcal{I}[x,u,\lambda] = \int_a^T \Big\{ f(t, x^\sigma(t), u^\sigma(t), x(T)) + \lambda^\sigma(t)\left[g(t, x^\sigma(t), u^\sigma(t), x(T)) - x^\Delta(t)\right] \Big\}\Delta t, \tag{26}$$

subject to

$$x(a) = \alpha \quad (x(T) \text{ free}). \tag{27}$$

Suppose that $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is a weak local minimizer for the problem (18)–(20). Then the triple $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{\lambda}(\cdot))$ should be a weak local minimizer for the problem (26)–(27). Using (25) in (26) we write the functional in the form

$$\mathcal{I}[x,u,\lambda] = \int_a^T [H(t, x^\sigma, u^\sigma, \lambda^\sigma, x(T)) - \lambda^\sigma(t) x^\Delta(t)]\Delta t. \tag{28}$$

Applying Theorem 3.2 to the problem (26)–(27), in view of (28), gives conditions (21)–(24). □

Remark 5. If $\mathbb{T}$ is a regular time scale, then by Theorem 3.3 the transversality condition (24) can be written in the form

$$\begin{aligned}\lambda^\sigma(\rho(T)) = \mu(\rho(T)) H_{x^\sigma}(\rho(T), x^\sigma(\rho(T)), u^\sigma(\rho(T)), \lambda^\sigma(\rho(T)), x(T))\\ + \int_a^T H_z(t, x^\sigma(t), u^\sigma(t), \lambda^\sigma(t), x(T))\Delta t.\end{aligned}$$

Example 5. Consider the problem

$$\begin{gathered}\text{minimize} \quad \mathcal{L}[x,u] = \int_0^3 (u^\sigma(t))^2 + t^2(x(3)-1)^2 \Delta t,\\ x^\Delta(t) = u^\sigma(t),\end{gathered} \tag{29}$$

$$x(0) = 0, \quad (x(3) \text{ free}). \tag{30}$$

To find candidate solutions for the problem, we start by forming the Hamiltonian function

$$H(t, x^\sigma, u^\sigma, \lambda^\sigma, x(3)) = (u^\sigma)^2 + t^2(x(3)-1)^2 + \lambda^\sigma u^\sigma.$$

Candidate solutions $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ are those satisfying the following conditions:

$$(\lambda^\sigma(t))^\Delta = 0, \tag{31}$$

$$u^\sigma(t) = x^\Delta(t), \quad x(0) = 0, \tag{32}$$

$$2u^\sigma(t) + \lambda^\sigma(t) = 0, \tag{33}$$

$$\lambda^\sigma(\rho(3)) = \int_0^3 2t^2(x(3)-1)\Delta t. \tag{34}$$

From (31)–(33) we conclude that $\tilde{x}(t) = ct$. In order to determine c we use the transversality condition (34) which we can write as

$$-2c = \int_0^3 2t^2(3c-1)\Delta t. \tag{35}$$

The value of the delta integral in (35) depends on the time scale. Let us compute, for example, this delta integral on $\mathbb{T} = \mathbb{Z}$ and on $\mathbb{T} = q^{\mathbb{N}_0}$ for $q = 2$. For $\mathbb{T} = \mathbb{Z}$

$$\int_0^3 2t^2(3c-1)\Delta t = \sum_{k=0}^{2} 2k^2(3c-1) = 10(3c-1). \tag{36}$$

Equations (35) and (36) yield $c = \frac{5}{16}$. Therefore, the extremal of the problem (29)–(30) on $\mathbb{T} = \mathbb{Z}$ is $\tilde{x}(t) = \frac{5}{16}t$. On the other hand, for $\mathbb{T} = 2^{\mathbb{N}_0}$ we have

$$\int_0^3 2t^2(3c-1)\Delta t = 2\,(1-3c) \sum_{t\in\{1,2\}} t^3 = 18(1-3c)\,,$$

and $\tilde{x}(t) = \frac{9}{26}t$.

When $\mathbb{T}=\mathbb{R}$ we immediately obtain from Theorem 3.4 the following corollary.

Corollary 4. *If $(\tilde{x}(\cdot),\tilde{u}(\cdot))$ is a solution of the problem*

$$\begin{gathered}\textit{minimize}\quad \mathcal{L}[x,u]=\int_a^T f(t,x(t),u(t),x(T))dt\\ x'(t)=g(t,x(t),u(t),x(T))\\ x(a)=\alpha\quad (x(T)\ \textit{free}),\end{gathered}$$

where α, a, $T\in\mathbb{R}$, $a<T$, then there exists a function $\tilde{\lambda}(\cdot)$ such that the triple $(\tilde{x}(\cdot),\tilde{u}(\cdot),\tilde{\lambda}(\cdot))$ satisfies the Hamiltonian system

$$x'(t)=H_\lambda\,,\quad \lambda'(t)=-H_x\,,$$

the stationary condition

$$H_u=0,$$

for all $t\in[a,T]$, and the transversality condition

$$\lambda(T)=\int_a^T H_z dt, \tag{37}$$

where the Hamiltonian H is defined by $H(t,x,u,\lambda,z)=f(t,x,u,z)+\lambda g(t,x,u,z)$.

Remark 6. In the classical context f and g do not depend on $x(T)$. In that case the transversality condition (37) coincides with the standard one ($\lambda(T)=0$) and Corollary 4 coincides with the Hestenes theorem [18] (a weak form of the Pontryagin maximum principle [26]). We were not able to find a single reference to the transversality condition (37) in the literature of optimal control.

We illustrate the use of Corollary 4 with an example:

Example 6. Consider the problem

$$\begin{gathered}\text{minimize}\quad \mathcal{L}[x,u]=\int_{-1}^1 (u(t))^2dt,\\ x'(t)=u(t)+x(1)t,\end{gathered}\tag{38}$$

$$x(-1)=1\quad (x(1)\text{ free}). \tag{39}$$

We begin by writing the Hamiltonian function

$$H(t,x,u,\lambda,x(1))=u^2+\lambda(u+x(1)t).$$

Candidate solutions $(\tilde{x}(\cdot),\tilde{u}(\cdot))$ are those satisfying the following conditions:

$$\lambda'(t)=0, \tag{40}$$

$$x'(t)=u(t)+x(1)t,\quad x(-1)=1, \tag{41}$$

$$2u(t)+\lambda(t)=0, \tag{42}$$

$$\lambda(1)=\int_{-1}^1 \lambda(t)t dt. \tag{43}$$

The equation (40) has solution $\tilde{\lambda}(t)=c$, $-1\le t\le 1$, which upon substitution into (43) yields

$$c=\int_{-1}^1 ct dt=0\,.$$

From the stationary condition (42) we get $\tilde{u}(t) = 0$. Finally, substituting the optimal control candidate back into (41) yields

$$\tilde{x}'(t) = \tilde{x}(1)t.$$

Integrating the latter equation with the initial condition $\tilde{x}(-1) = 1$, we obtain

$$\tilde{x}(t) = \frac{1}{2}\tilde{x}(1)t^2 + 1 - \frac{1}{2}\tilde{x}(1). \tag{44}$$

Substituting $t = 1$ into (44) we get $\tilde{x}(1) = 1$. Therefore, the candidate to solution for the problem (38)–(39) is $\tilde{x}(t) = \frac{1}{2}t^2 + \frac{1}{2}$.

In certain cases it is easy to show that the extremal candidates obtained from Theorem 3.2, Theorem 3.4, or one of the corollaries are indeed minimizers.

Theorem 3.5. *Let $(x^\sigma, u^\sigma, z) \to f(t, x^\sigma, u^\sigma, z)$ be jointly convex in (x^σ, u^σ, z) and $(x^\sigma, u^\sigma, z) \to g(t, x^\sigma, u^\sigma, z)$ be linear in (x^σ, u^σ, z) for each t. If $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{\lambda}(\cdot))$ is a solution of system (21)–(24), then $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is a global minimizer of (18)–(20).*

Proof. Since f is convex in (x^σ, u^σ, z) for any admissible pair $(x(\cdot), u(\cdot))$, we have

$$\begin{aligned}
\mathcal{L}[x,u] - \mathcal{L}[\tilde{x},\tilde{u}] &= \int_a^T \left[f(t, x^\sigma(t), u^\sigma(t), x(T)) - f(t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{x}(T))\right]\Delta t \\
&\geq \int_a^T \Big[f_{x^\sigma}(t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{x}(T))(x^\sigma(t) - \tilde{x}^\sigma(t)) \\
&\quad + f_{u^\sigma}(t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{x}(T))(u^\sigma(t) - \tilde{u}^\sigma(t)) \\
&\qquad + f_z(t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{x}(T))(x(T) - \tilde{x}(T))\Big]\Delta t.
\end{aligned}$$

Because the triple $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{\lambda}(\cdot))$ satisfies equations (22),(23), and (24), we obtain

$$\begin{aligned}
\mathcal{L}[x,u] - \mathcal{L}[\tilde{x},\tilde{u}] &\geq \int_a^T \Big[-\tilde{\lambda}^\sigma(t) g_{x^\sigma}(\cdots)(x^\sigma(t) - \tilde{x}^\sigma(t)) \\
&\quad - (\tilde{\lambda}^\sigma(t))^\Delta (x^\sigma(t) - \tilde{x}^\sigma(t)) - \tilde{\lambda}^\sigma(t) g_{u^\sigma}(\cdots)(u^\sigma(t) - \tilde{u}^\sigma(t)) \\
&\qquad - \tilde{\lambda}^\sigma(t) g_z(\cdots)(x(T) - \tilde{x}(T))\Big]\Delta t \\
&\qquad + \left[\tilde{\lambda}^\sigma(\rho(T)) - \int_{\rho(T)}^T H_{x^\sigma}(\cdots\cdot)\Delta t\right](x(T) - \tilde{x}(T)),
\end{aligned}$$

where $(\cdots) = (t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{x}(T))$ and $(\cdots\cdot) = \left(t, \tilde{x}^\sigma(t), \tilde{u}^\sigma(t), \tilde{\lambda}^\sigma(t), \tilde{x}(T)\right)$. Integrating by parts the term in $(\tilde{\lambda}^\sigma(t))^\Delta$ we get

$$\begin{aligned}
\mathcal{L}[x,u] - \mathcal{L}[\tilde{x},\tilde{u}] &\geq \int_a^T \Big[-\tilde{\lambda}^\sigma(t) g_{x^\sigma}(\cdots)(x^\sigma(t) - \tilde{x}^\sigma(t)) \\
&+ \tilde{\lambda}^\sigma(t)(x^\Delta(t) - \tilde{x}^\Delta(t)) - \tilde{\lambda}^\sigma(t) g_{u^\sigma}(\cdots)(u^\sigma(t) - \tilde{u}^\sigma(t)) - \tilde{\lambda}^\sigma(t) g_z(\cdots)(x(T) - \tilde{x}(T))\Big]\Delta t \\
&+ \left[\tilde{\lambda}^\sigma(\rho(T)) - \int_{\rho(T)}^T H_{x^\sigma}(\cdots\cdot)\Delta t\right](x(T) - \tilde{x}(T)) - \tilde{\lambda}^\sigma(T)(x(T) - \tilde{x}(T)).
\end{aligned}$$

But from (22) and properties of the delta integral, we have

$$\tilde{\lambda}^\sigma(T) = \tilde{\lambda}^\sigma(\rho(T)) - \int_{\rho(T)}^T H_{x^\sigma}(\cdots\cdot)\Delta t\,.$$

Rearranging the terms we obtain:

$$\begin{aligned}\mathcal{L}[x,u]-\mathcal{L}[\tilde{x},\tilde{u}] \geq \int_a^T \Big\{ & -\tilde{\lambda}^\sigma(t)\big[\tilde{x}^\Delta(t)-x^\Delta(t)+g_{x^\sigma}(\cdots)(x^\sigma(t)-\tilde{x}^\sigma(t)) \\ & + g_{u^\sigma}(\cdots)(u^\sigma(t)-\tilde{u}^\sigma(t))+g_z(\cdots)(x(T)-\tilde{x}(T))\big]\Big\}\Delta t. \qquad (45)\end{aligned}$$

The intended conclusion follows from (21) and linearity of g in (x^σ, u^σ, z): the right hand side of inequality (45) is equal to zero, that is,

$$\mathcal{L}[x,u] \geq \mathcal{L}[\tilde{x},\tilde{u}]$$

for each admissible pair $(x(\cdot), u(\cdot))$. □

Example 7. Consider the problem (38)–(39) in Example 6. The integrand is independent of (x,z) and convex in u. The right-hand side of the control system is linear in (u,z) and independent of x. Hence, Theorem 3.5 asserts that the extremal

$$\tilde{x}(t) = \frac{1}{2}\left(t^2+1\right)$$
$$\tilde{u}(t)=0, \quad \tilde{\lambda}(t)=0,$$

found in Example 6 gives the global minimum to the problem.

Example 8. Consider again the problem from Example 4. Replacing x^Δ by u^σ we can rewrite problem (13)–(14) in the following form:

$$\text{minimize} \quad \mathcal{L}[x] = \int_0^1 \left(\sqrt{1+(u^\sigma(t))^2}+\beta(x(1)-1)^2\right)\Delta t$$

subject to

$$x^\Delta(t)=u^\sigma(t), \quad x(0)=0 \quad (x(1) \text{ free}).$$

The function f is independent of x and convex in (u,z). The right-hand side of the control system is linear in u and independent of (x,z). Therefore, by Theorem 3.5 $\tilde{x}(t)=\alpha(\beta)t$ is the global minimizer for the problem.

Acknowledgements. The first author was supported by Białystok Technical University, via a project of the Polish Ministry of Science and Higher Education "Wsparcie miedzynarodowej mobilnosci naukowcow"; the second author by the R&D unit CEOC, via FCT and the EC fund FEDER/POCI 2010. ABM is grateful to the good working conditions at the Department of Mathematics of the University of Aveiro where this research was carried out.

REFERENCES

[1] R. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26.
[2] C. D. Ahlbrandt, M. Bohner and J. Ridenhour, *Hamiltonian systems on time scales*, J. Math. Anal. Appl. **250** (2000), no. 2, 561–578.
[3] C. D. Ahlbrandt and C. Morian, *Partial differential equations on time scales*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 35–55.
[4] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, J. Vib. Control, in press, arXiv:0811.3650
[5] F. M. Atici, D. C. Biles and A. Lebedinsky, An application of time scales to economics, Math. Comput. Modelling **43** (2006), no. 7-8, 718–726.
[6] F. M. Atici and F. Uysal, A production-inventory model of HMMS on time scales, Appl. Math. Lett. **21** (2008), no. 3, 236–243.

[7] B. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, in *Qualitative theory of differential equations (Szeged, 1988)*, Colloq. Math. Soc. János Bolyai, 53, North-Holland, Amsterdam, 1990, 37–56.
[8] G. Bangerezako, *Variational q-calculus*, J. Math. Anal. Appl. **289** (2004), no. 2, 650–665.
[9] Z. Bartosiewicz and D. F. M. Torres, *Noether's theorem on time scales*, J. Math. Anal. Appl. **342** (2008), no. 2, 1220–1226.
[10] M. Bohner, *Calculus of variations on time scales*, Dynam. Systems Appl. **13** (2004), no. 3-4, 339–349.
[11] M. Bohner and G. Sh. Guseinov, *Double integral calculus of variations on time scales*, Comput. Math. Appl. **54** (2007), no. 1, 45–57.
[12] M. Bohner and A. Peterson, "Dynamic equations on time scales", Birkhäuser Boston, Boston, MA, 2001.
[13] M. Bohner and A. Peterson, "Advances in dynamic equations on time scales", Birkhäuser Boston, Boston, MA, 2003.
[14] T. Ernst, *The different tongues of q-calculus*, Proc. Estonian Acad. Sci. **57** (2008), no. 2, 81–99.
[15] R. A. C. Ferreira and D. F. M. Torres, *Remarks on the calculus of variations on time scales*, Int. J. Ecol. Econ. Stat. **9** (2007), no. F07, 65–73.
[16] R. A. C. Ferreira and D. F. M. Torres, *Higher-order calculus of variations on time scales*, in "Mathematical Control Theory and Finance", Springer, Berlin, 2008, pp. 149–159.
[17] I. M. Gelfand and S. V. Fomin, "Calculus of variations", Prentice Hall, Englewood Cliffs, N.J., 1963.
[18] M. R. Hestenes, "Calculus of variations and optimal control theory", Wiley, New York, 1966.
[19] R. Hilscher and V. Zeidan, *Calculus of variations on time scales: weak local piecewise* $C^1_{\rm rd}$ *solutions with variable endpoints*, J. Math. Anal. Appl. **289** (2004), no. 1, 143–166.
[20] R. Hilscher and V. Zeidan, *Weak maximum principle and accessory problem for control problems on time scales*, Nonlinear Analysis: Theory, Methods & Applications, in press, DOI: 10.1016/j.na.2008.04.025.
[21] F. H. Jackson, *q-Difference Equations*, Amer. J. Math. **32** (1910), no. 4, 305–314.
[22] V. Kac and P. Cheung, "Quantum calculus", Springer, New York, 2002.
[23] W. G. Kelley and A. C. Peterson, "Difference equations", Academic Press, Boston, MA, 1991.
[24] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, "Dynamic systems on measure chains", Kluwer Acad. Publ., Dordrecht, 1996.
[25] A. B. Malinowska and D. F. M. Torres, *Necessary and sufficient conditions for local Pareto optimality on time scales*, J. Math. Sci. (N. Y.), 2009, in press, arXiv:0801.2123
[26] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, "The mathematical theory of optimal processes", Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

E-mail address: abmalinowska@ua.pt
E-mail address: delfim@ua.pt